Loschmidt echo in a system of interacting electrons

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(Dated: February 1, 2008)

We study the Loschmidt echo for a system of electrons interacting through mean-field Coulomb forces. The electron gas is modeled by a self-consistent set of hydrodynamic equations. It is observed that the quantum fidelity drops abruptly after a time that is proportional to the logarithm of the perturbation amplitude. The fidelity drop is related to the breakdown of the symmetry properties of the wave function.

Introduction.—In a famous controversy with L. Boltzmann, J. Loschmidt pointed out that, if one reverses the velocities of all particles in a physical system, the latter would evolve back to its initial state, thus violating the second law of thermodynamics. The main objection to this argument is that velocity reversal is a very unstable operation and tiny errors in the reversal quickly restore normal entropy increase.

More recently, the original idea of Loschmidt was revived in the context of quantum information theory. Indeed, any attempt at coding information using quantum bits is prone to failure if a small coupling to an uncontrollable environment destroys the unitary evolution of the wave function (decoherence) [1]. In order to estimate the robustness of the system against perturbations from the environment, the following procedure has been suggested. The system is allowed to evolve under the action of an unperturbed Hamiltonian until time T; then it is evolved backwards in time until $2T$ with the original Hamiltonian plus a small perturbation (the 'environment'). The square of the scalar product of the initial and final states defines the quantum fidelity of the system (Loschmidt echo) and has been the object of intense study in recent years. Jalabert and Pastawski [2] have proven that, for perturbations that are classically weak but quantum-mechanically strong, the fidelity decay rate only depends on the classical Lyapunov exponent of the unperturbed system. This universal behavior was later corroborated by numerical simulations [3, 4] and experiments [5]. For weaker perturbations, the decay rate is still exponential, but perturbation-dependent (Fermi golden rule regime). For still weaker perturbations, the decay is Gaussian [3].

An equivalent approach to the Loschmidt echo was proposed earlier by Peres [6]. In order to study the separation of classical trajectories, it is customary to compare the evolution of two slightly different initial conditions. Peres noted that one could just as well compare the *same* initial condition evolving in two slightly different Hamiltonians, an unperturbed one H_0 and a perturbed one $H = H_0 + \delta H$. The fidelity at time t is then defined as the square of the scalar product of the wave functions evolving with H_0 and H respectively: $F(t) = \left|\langle \psi_{H_0}(t) | \psi_H(t) \rangle\right|^2$. The latter approach is the one adopted throughout the present paper.

Virtually all theoretical investigations of the Loschmidt echo consider one-particle systems evolving in a given (usually chaotic) Hamiltonian. The aim of the present work is to explore the more realistic case of a system of many interacting particles, particularly electrons. In order to obtain a tractable model, we shall assume that the electrons interact via the electrostatic mean field, their dynamics being described by a set of one-dimensional (1D) hydrodynamic equations. As many experimental studies on quantum information involve the manipulation of charged particles, our approach may shed some light on the robustness of such systems against perturbations from the environment.

Model.—The physical properties of our model are best illustrated by considering its classical counterpart, the so-called 'cold plasma' model [7, 8]. In the latter, the electron population is described by a phase-space distribution function of the type: $f(x,v,t) = n(x,t)\,\delta[v-u(x,t)]$, where n and u are respectively the electron density and average velocity, and δ denotes the Dirac delta function. The support of such a distribution function in the 2D phase space (x,v) is a 1D curve defined by the relation $v = u(x,t)$. The electron distribution evolves according to the (collisionless) Vlasov equation. The ions are motionless with uniform equilibrium density n_0 and periodic boundary conditions (with $-L/2 \le x \le L/2$) are assumed for all variables.

As long as there is no particle overtaking in the phase space, each position x corresponds to a well-defined velocity $u(x,t)$. In this case, the Vlasov equation can be reduced to a closed set of pressureless hydrodynamic equations:

$$\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{e}{m}\frac{\partial \phi}{\partial x}, \tag{2}$$

where $-e$ an m are respectively the electron charge and mass, and $\phi(x,t)$ is the electric potential obeying Poisson's equation

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\varepsilon_0}(n - n_0). \tag{3}$$

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When particles overtake each other, the function $x \to u(x)$ becomes multivalued. The above hydrodynamic description then breaks down, although the microscopic Vlasov model remains valid.

A trivial stationary solution of Eqs. (1)-(2) is given by $n = n_0$, $u = 0$. The electron dynamics can be excited by modulating the initial velocity: $u(x, t = 0^+) = V_0 \cos(k_0 x)$, with $k_0 = 2\pi/L$, which is equivalent to applying an instantaneous electric field at time $t = 0$. The classical dynamics is determined by a single dimensionless parameter, namely the normalized wave number of the initial perturbation $K_0 = k_0 V_0/\omega_p$, where $\omega_p = (e^2 n_0/m\varepsilon_0)^{1/2}$ is the electron plasma frequency. Note that K_0 can be viewed as the ratio of kinetic to potential (electric) energy. It can be shown that two different regimes exist. When $K_0 < 1$, electric repulsion dominates, so that the electrons never overtake each other in the phase space. In this case, the hydrodynamic equations (1)-(2) can be solved analytically and the solution displays nonlinear oscillations at the plasma frequency. When $K_0 > 1$, the analytical solution breaks down and the dynamics must be described by the microscopic Vlasov equation. In the extreme case $K_0 \gg 1$, the dynamics becomes again integrable, because it reduces to that of free-streaming electrons. For moderate values of K_0 (but still larger than unity), the electrons can be alternately free streaming and trapped by the self-consistent (SC) potential. This regime corresponds to the formation of complex vortices in the phase space and leads to a chaotic dynamics, as was pointed out for the similar scenario of nonlinear Landau damping [9]. In this work, we will be mainly interested in the chaotic regime and use the value $K_0 = 2$.

Quantum corrections to the hydrodynamic equations (1)-(2) were previously derived [10]. For fermions at zero temperature (a case that is relevant to electrons in metals), the momentum conservation equation (2) should be modified as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{e}{m}\frac{\partial \phi}{\partial x} + \frac{\hbar^2}{2m^2}\frac{\partial}{\partial x}\left(\frac{\partial_x^2\sqrt{n}}{\sqrt{n}}\right) - \frac{1}{mn}\frac{\partial P_F}{\partial x}. \quad (4)$$

The second term on the right-hand side is the Bohm potential: this is a dispersive term that prevents the breakdown of the quantum hydrodynamics even when $K_0 > 1$ [8]. The third term is the Fermi pressure, which in 1D can be written as: $P_F/P_0 = (n/n_0)^3$, where the equilibrium pressure is given by the usual formula, $P_0 = \frac{2}{5} n_0 E_F$. E_F is the Fermi energy computed with the equilibrium density.

The continuity equation (1) and the quantum momentum equation (4) can be written in the form of a single nonlinear Schrödinger equation by introducing the effective wave function $\Psi(x,t) = \sqrt{n(x,t)}\exp(iS(x,t)/\hbar)$, where $S(x,t)$ is defined according to the relation $mu = \partial_x S$, and $n = |\Psi|^2$. The wave function Ψ obeys the equation

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} - e\phi\Psi + \frac{3}{5}E_F\frac{|\Psi|^4}{n_0^2}\Psi. \quad (5)$$

Equation (5) with Poisson's equation (3) constitute the mathematical model used throughout this Letter. The equilibrium Hamiltonian H_0 is time dependent, as it depends self-consistently on the wave function, but conserves both the total mass and the total energy. The initial condition is analog to the classical one described in the preceding paragraphs and can be easily derived from the velocity perturbation $u(x, t = 0^+)$ by using the relation between S and u. Two more dimensionless parameters (in addition to K_0) intervene in the quantum model: (i) the normalized Planck constant $h = \hbar\omega_p/mV_0^2$, which measures the importance of quantum effects; and (ii) the normalized Fermi velocity v_F/V_0. The latter affects very little the results (provided it is not too large), and will be fixed at $v_F/V_0 = 0.1$ in the forthcoming simulations.

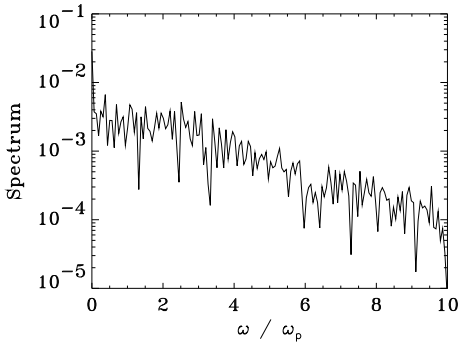

FIG. 1: Frequency spectrum of the potential energy, for an unperturbed evolution with $K_0 = 2$ and $h = 0.05$.

The numerical solution of Eq. (5) is obtained through a splitting scheme that separates the kinetic and potential parts of the Hamiltonian. Derivatives are computed with centered differences. The resulting algorithm is second-order accurate both in space and time.

Results.—First, we characterize the spectral properties of the unperturbed Hamiltonian. We consider a case with $K_0 = 2$ and $h = 0.05$ and plot in Fig. 1 the frequency spectrum of the time history of the electrostatic energy. The spectrum is broad and virtually flat in the range $0 < \omega \lesssim 3\omega_p$. The dynamics is therefore sufficiently irregular to enable us to compare our results to those obtained for a single-particle chaotic Hamiltonian.

In order to study the behavior of the quantum fidelity, we need to compare the evolution of Ψ obtained with the unperturbed and perturbed Hamiltonians. We use a static perturbation consisting of a sum of a large number of uncorrelated waves: $\delta H(x)/mV_0^2 = \epsilon\sum_{j=N_{\min}}^{N_{\max}}\cos(k_j x + \alpha_j)$, where ϵ is the amplitude of the perturbation, $k_j = j\ (2\pi/L)$, and the α_j are random phases. The wave number spectrum of the perturbation (i.e. the values of $N_{\min}$ and $N_{\max}$) affect only very weakly the behavior of the fidelity: therefore, we will focus our analysis on the dependence of the fidelity on the amplitude ϵ.

A typical result for the quantum fidelity is presented in Fig. 2. Contrarily to most single-particle cases, the

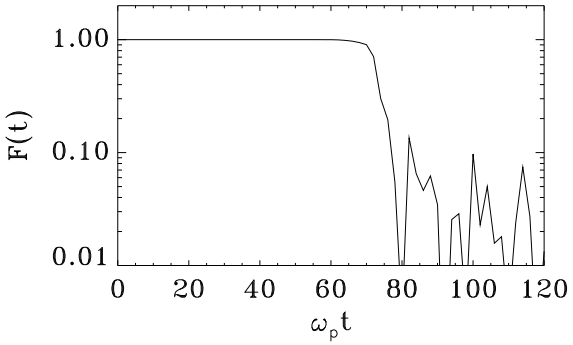

FIG. 2: Fidelity decay for $K_0 = 2$, $h = 0.05$, and perturbation $\epsilon = 10^{-9}$.

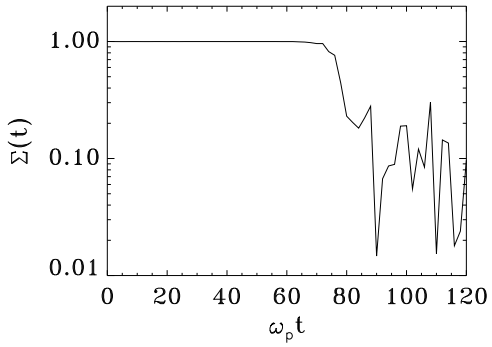

FIG. 3: Evolution of the symmetry $\Sigma(t)$ for the same case as in Fig. 2.

fidelity does not decay exponentially. Instead, it remains equal to unity until a critical time τ_c, after which it decays abruptly within a few units of ω_p^{-1}. This behavior is generic and was observed for all set of parameters that were studied, provided the dynamics is sufficiently irregular. Numerical tests showed that the value of the critical time is independent on the time step. However, as the underlying dynamics is chaotic, evolutions computed using different time steps will inevitably diverge for long times. Therefore, the details of the evolution for $t \gg \tau_c$ are not quantitatively meaningful, and simply indicate that the fidelity has dropped to very small values.

The observed drop in the quantum fidelity is related to a sudden symmetry breaking of the wave function. Indeed, the evolution equations (3)-(5) for the unperturbed Hamiltonian are invariant under the transformation $x \to -x$. If the initial condition is an even function of x, this symmetry is thus preserved in time, i.e. $\Psi(x,t) = \Psi(-x,t)$, $\forall t$. But the perturbation δH possesses no particular symmetry, and one would expect that the symmetry of the initial state quickly deteriorates. The symmetry properties can be conveniently measured by the following quantity:

$$\Sigma(t) = \frac{2}{n_0 L} \left| \int_0^{L/2} \Psi(x,t)\ \Psi^\star(-x,t)\ dx \right|^2 , \qquad (6)$$

which is equal to unity when $\Psi(x,t) = \Psi(-x,t)$. The evolution of $\Sigma(t)$ for the perturbed Hamiltonian is plotted in Fig. 3. The drop of the quantum fidelity happens virtually at the same time as the breaking of the wave function symmetry. This behavior is also generic across a wide range of parameters.

We further investigated the dependence of the critical time τ_c on the perturbation amplitude ϵ, for various values of the normalized Planck constant h. The critical time is defined as the time at which the fidelity has dropped to 10% of its maximum value, i.e. $F(\tau_c) = 0.1$. Figure 4 shows that, for small values of h, τ_c depends logarithmically on the perturbation amplitude, i.e. $\tau_c \sim -t_0 \ln \epsilon$, with $\omega_p t_0 \simeq 4.3$ (this is the straight line depicted in Fig. 4). This logarithmic dependence appears to be universal (both in slope and absolute value), at least for the values of K_0 and v_F/V_0 adopted in these runs. For larger values of Planck's constant ($h \gtrsim 0.2$), this behavior is less neat, particularly for small perturbations.

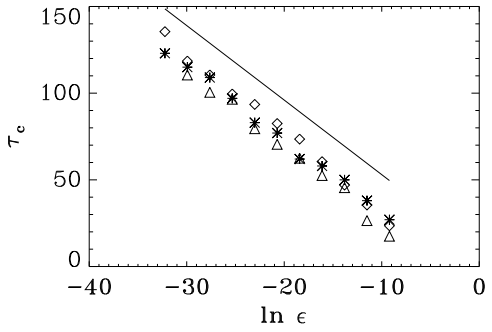

FIG. 4: Critical time τ_c (in units of ω_p^{-1}) versus perturbation amplitude ϵ, for $h = 0.05$ (stars), $h = 0.025$ (diamonds), and $h = 0.0125$ (triangles). The solid line represents the curve $\tau_c \sim -t_0 \ln \epsilon$, with $\omega_p t_0 = 4.3$.

A similar pattern was observed for a chaotic quantum map [11]: in that case, the fidelity stays equal to unity until a critical time, after which it starts to decay exponentially at the classical Lyapunov rate. No sudden drop was observed, as is the case for our simulations.

In order to better evaluate the impact of the SC field, we performed some simulations where all nonlinear terms have been suppressed in Eqs. (3)-(5). The first nonlinearity comes from the Fermi pressure and can be removed simply by setting $E_F = 0$. The SC nonlinearity comes from the fact that the electric potential depends on the wave function through the electron density $n = |\Psi|^2$. To remove this nonlinearity, we define the electron density independently of Ψ, as the sum of traveling plane waves: $n = n_{\rm ext} \equiv n_0[1 + \delta \sum_{j=1}^{25} k_j^2 \cos(k_j x - \omega_p t + \alpha_j')]$, where δ is the amplitude of the density fluctuations, and the α_j' are random phases. This definition is plugged into Poisson's equation to yield the electric potential. As the resonances of the plane waves overlap in phase space, the resulting (time-dependent) Hamiltonian H_0 is likely to display chaotic regions [12]. The fidelity decay is studied

by perturbing the Hamiltonian in the same way as in the SC case.

In Fig. 5 we plot the quantum fidelity for $\delta = 0.5$, $h = 0.025$, and several values of the perturbation ϵ. The fidelity decay is exponential and begins at $t = 0$. The decay rate is approximately proportional to the square of the perturbation, which shows that we are in the so-called Fermi golden rule regime [3, 11]. However, contrarily to Ref. [11], no plateau was observed for short times.

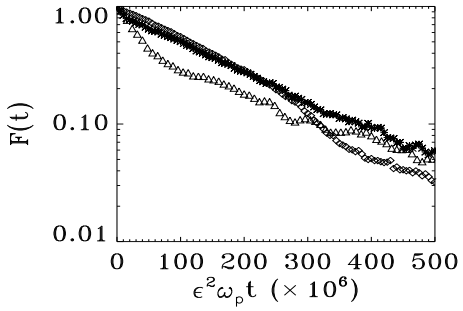

FIG. 5: Fidelity decay for a purely external Hamiltonian: $\epsilon = 10^{-3}$ (stars); $\epsilon = 2{\times}10^{-3}$ (triangles); $\epsilon = 5{\times}10^{-4}$ (diamonds). Time is rescaled to the square of the perturbation ϵ.

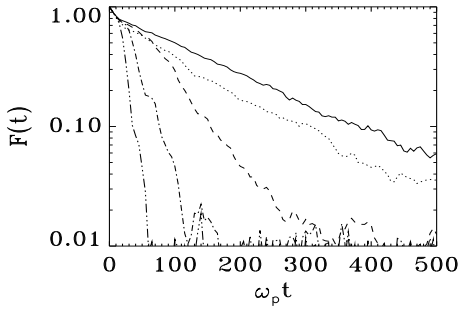

FIG. 6: Fidelity decay for a mixed external and SC Hamiltonian, for $\beta = 0$ (solid line); $\beta = 0.01$ (dotted); $\beta = 0.03$ (dashed); $\beta = 0.1$ (dot-dashed); $\beta = 0.3$ (dot-dot-dashed).

Finally, we studied a case where both the SC and the external fields are present. This is accomplished by defining the electron density as: $n(x,t) = n_{\rm ext} + \beta(|\Psi|^2 - n_0)$. By varying β and δ, we can move continuously from a purely 'external' regime ($\beta = 0$) to a purely SC one ($\delta = 0$, $\beta = 1$). We concentrate on a case with $\delta = 0.5$, $h = 0.025$, and $\epsilon = 10^{-3}$, and vary the value of β (Fig. 6). For short times, the fidelity decays exponentially with the same rate as in the purely 'external' case. Subsequently, the decay becomes faster, and for large values of β we almost recover the abrupt drop of Fig. 2.

Discussion.— The present work is a first attempt at studying quantum fidelity decay in a system of electrons interacting through their SC electric field. Our numerical results show that the quantum fidelity can display a rapid decrease. Such effect is probably related to the fact that the unperturbed Hamiltonian H_0 depends on the wave function. When the perturbation δH induces a small change in Ψ, H_0 is itself modified, which in turns affects Ψ, and so on. Because of such nonlinear loop, the perturbed and unperturbed solutions can diverge very fast (typically, within a few ω_p^{-1}). In contrast, for the single-particle dynamics, H_0 is fixed and the solutions only diverge because of the small perturbation δH.

In summary, it appears that the 'natural' response of a SC system to a perturbation is a sudden drop of the fidelity after a quiescent period rather than an exponential decay. Further studies will be needed to understand whether these results extend to more complex models, going beyond the mean-field approach adopted here.

We acknowledge fruitful discussions with R. A. Jalabert.

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